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Tensor tomography of stresses in cubic single crystals

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Abstract

The possibility of optical tomography applying to investigation of a two-dimensional and a three-dimensional stressed state in single cubic crystals has been studied. Stresses are determined within the framework of the Maxwell piezo-optic law (linear dependence of the permittivity tensor on stresses) and weak optical anisotropy. It is shown that a complete reconstruction of stresses in a sample is impossible both by translucence it in the parallel planes system and by using of the elasticity theory equations. For overcoming these difficulties, it is offered to use a method of magnetophotoelasticity.

Keywords: Internal stresses; Polarization tomography; Integrated photoelasticity; Tensor of stresses.

1. Introduction

Vector and tensor field tomography makes a number of new and interesting nondestructive methods possible: polarimetric tomography of magnetic field in the tokamak plasma, measuring of electric field distribution in dielectric liquids on the basis of optical Kerr effect tomography, tomography of fluid flow. Interest in optical tensor field tomography has been simulated primarily by the possibility of applying integrated photoelasticity to the stress analysis of apparent models [1]. Residual stress is one of the most important characteristics of glass articles from the point of view of their strength and resistance [2]. In the case of optical glass, birefringence due to the residual stresses characterizes the optical quality of the article [3].

Integrated photoelasticity refers to polarization–optical methods of experimental mechanics that use tomographic measurement techniques. These methods measure the variation in parameters of polarized beam transmitted by a model under study [4]. In certain cases distribution of some stress components can be determined using this integrated optical information. Generally, it is impossible to achieve a complete reconstruction of stresses by translucence it in the system of parallel planes and using the equations of the elasticity theory [5]. The magnetophotoelasticity method (MPE) for overcoming this difficulty is to use additional magnetic field [6,7]. In this new measurement method, the model under study is subjected to a homogeneous magnetic field and is irradiated by light propagating along this field. The polarization plane of the light beam rotates in the sample due to the Faraday effect. Until today MPE has been used for the measurement of bending stresses in plates [8] and residual stresses in glass plates [9]. Tomographic application of MPE method is based

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on the exponential Radon transform of vector and tensor fields [10,11]. Algorithms for the reconstruction of the attenuated vectorial Radon transform [12,13] give an opportunity for investigation in nonhomogeneous magnetic field.

Polarization–optical methods are commonly used for investigation of physical properties of crystals [14,15]. Two-dimensional photoelasticity permits easy determination of stresses (which are constant through the thickness) in crystal plates [16]. Compared with isotropic objects, the application of the integrated photoelasticity becomes much more difficult in the case of single crystals due to the occurrence of natural anisotropy [17]. The first difficulty is associated with non-coincidence of the quasi-principal directions of the stress tensor and the permittivity tensor [18]. The second difficulty is connected with the problem in the theory of elasticity which is described in [19,24]. In the case of plane elastic strain residual stresses in a cubic single crystal can be reconstructed completely by using the method of the integrated photoelasticity [20].

The aim of this paper is to generalize the results of the parametric tensor tomography to the case of cubic single crystals.

2. Integrated photoelasticity

In integrated photoelasticity method the specimen is immersed in immersion bath and a beam of polarized light is passed through the specimen. In stress-free situation a cubic single crystal is optically isotropic and birefringence due to the residual stresses can be measured by the tomographic method. In the general case, the direct problem of light propagation in an inhomogeneous anisotropic medium is rather complicated, but in case of weakly birefringent media it is simplified.

We will introduce an orthogonal system of coordinates x, y, z and direct the axes of this system along the [1 0 0], [0 1 0] and [0 0 1] crystallographic directions. Additionally, we will introduce an orthogonal system of coordinates s, t, z which is rotated with respect to the initial system by an angle Θ in the plane x, y. Direction of translucence coincides with the direction of the t-axis. Propagation of polarized light through a weakly birefringent media is governed by the following equations [4,6,7]:

$$\frac{dE}{dt} = -iCPE, \qquad C = \frac{\omega}{2c\sqrt{\chi}}, \tag{1}$$

where

$$E = \begin{pmatrix} E_z \\ E_s \end{pmatrix},$$

$$P = \begin{bmatrix} \frac{1}{2}(\chi_{zz} - \chi_{ss}) & \chi_{zs} \\ \chi_{zs} & \frac{1}{2}(\chi_{ss} - \chi_{zz}) \end{bmatrix}.$$

Here E_z, E_s denote the components of the electric vector, c is the light speed, ω is the frequency, χ is permittivity of the stress free media, χ_{zz}, χ_{zs}, χ_{ss} are components of the permittivity tensor induced by the residual stresses. The matriciant of Eq. (1) (Jones matrix) $\Omega(\gamma, \alpha_0, \alpha_*)$ can be expressed via its characteristic parameters: γ is the characteristic phase difference, α_0 is the initial characteristic direction, α_* is the secondary characteristic direction. In the case of a slow rotation of quasi-principal directions along the light propagation direction these parameters are related to the components of the dielectric tensor through the relationships

$$2\gamma\cos(\alpha_0 + \alpha_*) = C\int(\chi_{ss} - \chi_{zz})dt = T_1(s, \theta);$$

$$\gamma\sin(\alpha_0 + \alpha_*) = C\int(\chi_{zs})dt = T_2(s, \theta).$$

Equations of photoelasticity of a single cubic crystal [14] are the following:

$$\chi_{ss} - \chi_{zz} = e_1\sigma_{ss} + e_2\sigma_{tt} + e_3\sigma_{zz} + e_6\sigma_{st}; \qquad \chi_{sz} = \pi_{44}\sigma_{sz}; \tag{2}$$

they contain the fourth-order elastic–optical tensor π_{ik}

$$e_1 = (\pi_{11} - \pi_{12}) - (\pi_{11} - \pi_{12} - \pi_{44})\frac{\sin^2(2\theta)}{2},$$

$$e_2 = (\pi_{11} - \pi_{12} - \pi_{44})\frac{\sin^2(2\theta)}{2},$$

$$e_3 = -(\pi_{11} - \pi_{12}),$$

$$e_6 = -(\pi_{11} - \pi_{12} - \pi_{44})\frac{\sin(4\theta)}{2}.$$

The essential difference of a problem under study from a problem for an isotropic medium is the presence of the second addend σ_{tt} in the Eq. (2). This fact considerably complicates the solution of the problem. At first, an algorithm of determining the residual stress in long cubic crystal (the assumption of plane deformation) will be presented. Then, the application of the parametric tomography to the reconstruction of stresses in the common case will be considered.

3. Reconstruction of stresses in case of plane strain deformation

A cylindrical crystal without axial stress gradient is illuminated in the plane $z = \text{const}$. The components σ_{sz}

and σ_{tz} are equal to zero and there is no rotation of the quasi-principal directions of the permittivity tensor. Thus, the problem of the optical tomography is simplified and only characteristic phase differences

$$2\gamma = \int E_1\sigma_{ss} + E_2\sigma_{tt} + E_3\sigma_{zz} + E_6\sigma_{st}dt = T_1(s,\theta),$$
$$E_i = Ce_i \tag{3}$$

are measured on the ray.

The optical relation (3) is complemented by equations of the elasticity theory: equations of state, equilibrium equations, compatibility equations. The residual stresses are considered to be of thermal character:

$$\begin{aligned}\sigma_{xx} &= C_{11}\varepsilon_{xx} + C_{12}(\varepsilon_{yy} + \varepsilon_{zz}) - \alpha T,\\ \sigma_{yy} &= C_{11}\varepsilon_{yy} + C_{12}(\varepsilon_{xx} + \varepsilon_{zz}) - \alpha T,\\ \sigma_{zz} &= C_{11}\varepsilon_{zz} + C_{12}(\varepsilon_{xx} + \varepsilon_{yy}) - \alpha T,\\ \sigma_{ik} &= 2C_{44}\varepsilon_{ik}, \quad i \neq k, \quad i,k = x,y,z.\end{aligned} \tag{4}$$

Here C_{ik} are the coefficients of the elasticity, α is the coefficient of thermal expansion, T is fictitious temperature, and ε_{ik} are the components of the strains tensor.

In the case of the plane strain deformation $\varepsilon_{iz} = 0$, and from Eqs. (4) it follows

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \sigma_{zz}}{C_{11} - C_{12}}, \quad \varepsilon_{yy} = \frac{\sigma_{yy} - \sigma_{zz}}{C_{11} - C_{12}}, \quad \varepsilon_{xy} = \frac{\sigma_{xy}}{C_{44}}.$$

Inserting the components of strain into the equation of compatibility

$$\frac{\partial^2}{\partial y^2}\varepsilon_{xx} + \frac{\partial^2}{\partial x^2}\varepsilon_{yy} = 2\frac{\partial^2}{\partial x\partial y}\varepsilon_{xy}$$

and expressing the components of stress through the Airy function

$$\sigma_{ij} = \delta_{ij}\Delta F - \frac{\partial^2 F}{\partial x_i \partial x_y}$$

one can obtain

$$\frac{\partial^4}{\partial x^4}F + \frac{(C_{11} - C_{12})}{C_{44}}\frac{\partial^4}{\partial^2 x\partial^2 y}F + \frac{\partial^4}{\partial y^4}F = \Delta\sigma_{zz}.$$

This equation can be written in more suitable form [20]:

$$\Delta_1\Delta_2 F = \frac{k}{1+k}(\Delta_1 + \Delta_2)\sigma_{zz}, \tag{5}$$

where

$$\Delta_1 F = \left(\frac{\partial^2}{\partial x^2} + k\frac{\partial^2}{\partial y^2}\right)F, \quad \Delta_2 F = \left(k\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F,$$

and k is a solution of the equation

$$k^2 - \frac{C_{11} - C_{12}}{C_{44}}k + 1 = 0.$$

This equation is based on the fact that the Airy function F and its normal derivative $\partial F/\partial n$ are equal to zero in the load free conditions on the lateral surface.

If the distribution of σ_{zz} is described only by harmonic functions, the other stress components do not develop in a cylinder. The same situation takes place in the case of an isotropic cylinder [21].

General solution of Eq. (5) can be written as the sum of functions $F = F_1 + F_2$ which must satisfy the equation

$$\Delta_1\left(\Delta_2 F_2 - G_1 - \frac{k}{1+k}\sigma_{zz}\right) + \Delta_2\left(\Delta_1 F_1 - G_2 - \frac{k}{1+k}\sigma_{zz}\right) = 0, \tag{6}$$

where $\Delta_i G_i = 0$.

We can write Eq. (6) as a system of the following equations:

$$\Delta_2 F_2 = \frac{k}{1+k}\sigma_{zz} + G_1, \quad \Delta_1 F_1 = \frac{k}{1+k}\sigma_{zz} + G_2,$$
$$\Delta_1\Delta_2(F_2 - F_1) = \frac{k(k-1)}{1+k}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\sigma_{zz}.$$

The boundary conditions for F_i give the integral equation for the determination of the G_i

$$\iint_s G_{1a}(\sigma_{zz} + G_2)ds = 0,$$
$$\iint_s G_{2a}(\sigma_{zz} + G_1)ds = 0.$$

Here G_{1a}, G_{2a} are arbitrary solution of corresponding equations $\Delta_i G_{ia} = 0$.

At last the line integral (3) can be simplified by using the Airy function:

$$\int \sigma_{ss}dt = \int \frac{\partial^2}{\partial t^2}Fdt = \int \sigma_{st}dt = -\int \frac{\partial^2}{\partial s\partial t}Fdt = 0.$$

Thus, residual stresses can be determined from the partial solution of the Eq. (5) and the ray integral equation is

$$E_2\int \frac{\partial^2}{\partial s^2}Fdt + E_3\int \sigma_{zz}dt = T_1(s,t).$$

The numerical solution of this system of equations is not considered as it is connected with the peculiarities of the measurements.

4. Application of the MPE method to the full determination of stresses

In the case of an arbitrary distribution of internal stresses in a sample, the number of variables increases and the reconstruction problem becomes significantly more complicated. In this case it is impossible to achieve a complete reconstruction of stresses by using only the equations of the elasticity theory. We apply the additional homogeneous magnetic field along the light propagation direction and measure the angle of rotation of the polarization plane due to the Faraday effect. The following two integrals can be determined using the polarization measurements [6,7]:

$$\int (E_1\sigma_{ss} + E_2\sigma_{tt} + E_3\sigma_{zz} + E_6\sigma_{st})e^{\beta t}dt = T_1(s,\theta,\beta)$$

$$\int E_4\sigma_{zs}e^{\beta t}dt = T_2(s,\theta,\beta), \quad E_4 = C\pi_{44}.$$

Here, $\beta = VH$ and V is the Verdet constant, H is the intensity of the magnetic field strength. The components of stress tensor must satisfy the equilibrium equations. The solution of these equations can be expressed through the stress functions [7,22]. According to the Helmholtz theorem, the two-dimensional vector field defined over xy plane can be represented as a sum of an irrotational (potential) field and of a solenoidal one:

$$\sigma_{zx} = \frac{\partial}{\partial x}\tau + \frac{\partial^2}{\partial y\partial z}N, \quad \sigma_{zy} = \frac{\partial}{\partial y}\tau - \frac{\partial^2}{\partial x\partial z}N.$$

Similarly, a two-dimensional symmetric tensor field defined over a plane xy can be expressed in terms of the three potentials:

$$\begin{aligned}\sigma_{xx} &= -\frac{\partial}{\partial z}\tau + 2\frac{\partial^2}{\partial x\partial y}N + \frac{\partial^2}{\partial y^2}F,\\ \sigma_{yy} &= -\frac{\partial}{\partial z}\tau - 2\frac{\partial^2}{\partial x\partial y}N + \frac{\partial^2}{\partial x^2}F,\\ \sigma_{xy} &= -\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)N - \frac{\partial^2}{\partial x\partial y}F.\end{aligned} \tag{7}$$

The stress functions must satisfy the following equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\hat{o} = -\frac{\partial}{\partial z}\acute{o}_{zz},$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)N = 0.$$

Tomographic reconstruction of shear stresses σ_{zi} in the xy plane, based on the values of the path integral T_2, is reduced to determining a 2D vector field and, according to the Helmholtz theorem to finding its solenoidal N and potential τ components. The algorithm of reconstruction of these components in the case of the magnetophotoelasticity was given in [6,7,10] and, the more general case of the attenuated vectorial Radon transform was given in [12,13]. Once the potentials τ and N are known the other stresses can be reconstructed if we find F and σ_{zz}. Using representation (7), we can transform path integral $T_1(s,\theta,\beta)$ by integrating in parts:

$$T_1(s,\theta,\beta) = E_1\beta^2\int Fe^{i\beta t}dt + E_2\int\frac{\partial^2}{\partial s^2}Fe^{i\beta t}dt$$

$$+E_3\int\sigma_{zz}e^{i\beta t}dt + iE_6\beta\int\frac{\partial}{\partial s}Fe^{i\beta t}dt + T_0(s,\theta,\beta)$$

Here, $T_0(s, \theta, \beta)$ is the function, containing known potentials τ, and N. Thus, by using the measurements at $\beta = 0$, and $\beta \neq 0$, we have the system of the ray integrals, which allow one to fully recover the tensor field.

5. Summary

The majority of investigations in the tensor stress field tomography are devoted to isotropic articles. In this paper we demonstrate the opportunities and difficulties of the integrated photoelasticity in the case of cubic single crystals. Reconstruction of the residual stresses in both cases is connected with the solution of the system of equations. In the case of the isotropic model this system can be solved step by step. In the second case the same equations cannot be separated, and they should be solved simultaneously. One of the drawbacks of the magnetophotoelasticity is the need for very precise optical measurements since the Faraday effect is very small. The using of multiple reflections is one of the possibilities to overcome this difficulty [23].

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